Evaluation of Fourier Transforms

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INTRODUCTION

THE NATURE of the specific evaluation problem presented by the use of Fourier methods in connection with some practical situation may vary over a wide range of forms and in its detailed aspects. In this connection, we do not necessarily confine our attention to the process of real integration with which the conventional Fourier integrals are concerned since, as pointed out in the Introduction,[1] it may be found expedient to replace $j\omega$ by the complex variable s and use methods of complex integration. So long as the integrand, apart from the exponential factor, is rational, the evaluation process is straightforward in terms of Cauchy's well-known residue theorem. On the other hand, if irrational and transcendental functions are involved, we have branch points, Riemann surfaces, and essential singularities to consider, and the known methods of complex integration fail to provide a practical answer to the evaluation problem.

In situations of this sort, one must resort to approximate methods of integration. These are based upon the practical attitude that it is necessary only to be able to construct a sequence of solutions that are approximately valid within finite time intervals that are consecutive, and collectively embrace the region of interest. This objective is carried out through finding appropriate rational approximants to the irrational and/or transcendental integrands that yield solutions with acceptably small error over the pertinent time intervals. Saddle-point methods, and other similar and extended forms for carrying out such approximate integration procedures by analytic means, are not yet completely developed or documented and involve too extensive a discussion to be gone into here.[2] Instead, we shall discuss semigraphical numerical methods and related analog and digital computer procedures appropriate to the more commonly encountered evaluation needs.

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[1] E. A. Guilleman, "The Fourier integral—a basic introduction," p. 227; this issue.

[2] For examples of this kind of integration, see M. Cerrillo, "Transient Phenomena in Waveguides," Research Laboratory of Electronics, (MIT), Technical Report No. 33, January, 1948.

I. SEMIGRAPHICAL NUMERICAL METHODS

In these methods of evaluation, the sufficiency of the real part of the spectrum or system function is exploited in combination with a method of approximating the time function or spectrum function through representing an appropriately chosen higher derivative by an impulse train. The pertinent forms for the Fourier integral are obtained from the basic forms given by (15) and (16) in the Introduction,[1] through differentiation under the integral sign with respect to $j\omega$ or t, respectively, and separation of the results into even and odd components after the fashion that the integrals (24) and (26) are derived. If we denote the time function by $f(t)$ and write its complex transform as $F(j\omega) = F_1(\omega) + jF_2(\omega)$, then we obtain in this way the following family of relations between the time function and the real part of the spectrum function:

$$f(t) = \frac{2}{\pi}\int_0^\infty F_1(\omega)\cos\omega t d\omega;$$

$$F_1(\omega) = \int_0^\infty f(t)\cos\omega t dt \tag{1}$$

$$f(t) = \frac{-2}{\pi t}\int_0^\infty F_1^{(1)}(\omega)\sin\omega t d\omega;$$

$$F_1(\omega) = \frac{-1}{\omega}\int_0^\infty f^{(1)}(t)\sin\omega t dt \tag{2}$$

$$f(t) = \frac{-2}{\pi t^2}\int_0^\infty F_1^{(2)}(\omega)\cos\omega t d\omega;$$

$$F_1(\omega) = \frac{-1}{\omega^2}\int_0^\infty f^{(2)}(t)\cos\omega t dt \tag{3}$$

and so forth. The superscript on $f(t)$ or on $F_1(\omega)$ indicates the derivative of corresponding order. These relations are, of course, valid only so long as the integrals exist, but this question need not concern us at the moment. In the forms (1), (2), (3), differentiation of $F_1(\omega)$ is with respect to ω, not to $j\omega$.

As an illustration of how these relations are used, consider the trapezoidal real part function shown in the

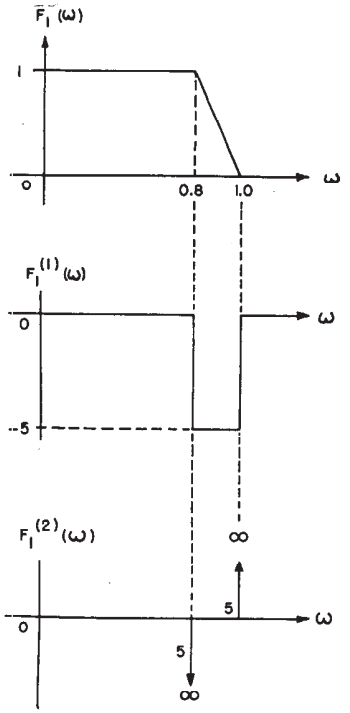

Fig. 1—Real part function reduced to impulses by differentiation.

top part of Fig. 1. Directly below it is drawn the first derivative and in the lower sketch we see that the second derivative has the impulse representation

$$F_1^{(2)}(\omega) = -5u_0(\omega - 0.8) + 5u_0(\omega - 1), \qquad (4)$$

in which $u_0(x)$ denotes a unit impulse occurring at $x=0$. Substitution into the first integral (3) gives by inspection

$$f(t) = \frac{10}{\pi t^2}\{\cos 0.8t - \cos t\}, \qquad (5)$$

having the initial value $f(0)=9/5\pi$ as it should, since this value by (1) equals $2/\pi$ times the net area under the real-part function.

Any $F_1(\omega)$ curve approximated by a broken line will have its second derivative given by a sum of impulses, and the first of the integrals (3) may then be used to write down at once the time function corresponding to this broken-line approximation of the spectrum function. The number of terms in the expression for the time function equals the number of breakpoints in the broken-line approximation. Since it is desirable to keep this number small, it is not always best to use a broken-line approximation for the function $F_1(\omega)$ itself. Thus if we apply the broken-line approximation to $F_1^{(1)}(\omega)$, then $F_1(\omega)$ becomes approximated by parabolic arcs, which yields a smooth result with fewer terms than the number of breakpoints required to get the same closeness of approximation out of a broken-line approximation applied directly to $F_1(\omega)$.

In connection with these thoughts, three things are of interest: the closeness with which $F_1(\omega)$ is approximated, the order of the derivative of $F_1(\omega)$ to which we apply the broken-line approximation, and the number of breakpoints or terms needed to represent $f(t)$. The following considerations enable us to interrelate these.

First, we remark that in these approximate methods of integration we always consider the function $F_1(\omega)$ to extend over a finite frequency interval whose width we shall normalize at unity, as is done in Fig. 1. The effect of chopping off an asymptote in order to achieve this representation, and the way in which one may nevertheless properly take account of the relationship between the behavior of $F(j\omega)$ for $(j\omega)\to\infty$ and that of $f(t)$ for $t\to 0$, will receive separate attention.

Imagine the νth derivative of $F_1(\omega)$ to be approximated by a broken line in such a way that the error (difference between the broken line and the actual curve) changes sign three times within each linear interval, which it normally will if a line is passed through each inflection point. Within the interval $0<\omega<1$ there will then be $3(n-1)$ changes of sign, if n equals the total number of break points in the piecewise linear approximation.[3] Hence there will be $3(n-1)/2$ cycles of error oscillation in the interval $0<\omega<1$. Let $3(n-1)/2=\tau/2\pi$, and assume a roughly sinusoidal error variation with amplitude ϵ in the piecewise linear approximation of the νth derivative. Then we have

$$\text{error in } F_1^{(\nu)}(\omega) \approx \epsilon \sin \omega\tau$$

$$\text{error in } F_1^{(\nu-1)}(\omega) \approx -\frac{\epsilon}{\tau_0}\cos \omega\tau$$

$$\text{error in } F_1^{(\nu-2)}(\omega) \approx -\frac{\epsilon}{\tau_0^2}\sin \omega\tau, \qquad (6)$$

and so forth. Hence, the approximate maximum error in the approximation of $F_1(\omega)$ itself becomes

$$\epsilon/\tau_0^{\nu} = \frac{\epsilon}{[3\pi(n-1)]^{\nu}}. \qquad (7)$$

Although somewhat crude in its derivation, this result is useful in estimating the error of approximation for particular values of n and ν. For example, if the first derivative of $F_1(\omega)$ is approximated in a piecewise linear manner involving four break points and a maximum error of 10 per cent, then (7) for $\nu=1$, $n=4$, and $\epsilon=0.1$

[3] If the piecewise linear approximated function is odd so that $\omega=0$ is not a break point, then the number of linear intervals equals n, but we prefer in this error estimation process to be conservative, and pick for n a larger rather than a smaller value in relation to the total number of cycles of error oscillation within the interval $0<\omega<1$.

shows that $F_1(\omega)$ will be approximated by parabolic arcs with a maximum error of less than one-half of one per cent, and the corresponding time function will be represented by four terms.

If we observe that the function $F_1(\omega)$ is even but that its derivatives of odd order are odd, then a piecewise linear approximation of the *vth* derivative with n break points is seen to yield for the derivative of order $(\nu+2)$ the impulse representation

$$F_1^{(\nu+2)}(\omega) = \sum_{k=1}^{n} a_k[u_0(\omega - \omega_k) + (-1)^\nu u_0(\omega + \omega_k)]. \quad (8)$$

The corresponding time function is found to be

$$f(t) = \frac{-2(-1)^{\nu/2}}{\pi t^{\nu+2}} \sum_{k=1}^{n} a_k \cos \omega_k t \quad \text{for } \nu \text{ even}, \quad (9)$$

and

$$f(t) = \frac{2(-1)^{(\nu-1)/2}}{\pi t^{\nu+2}} \sum_{k=1}^{n} a_k \sin \omega_k t \quad \text{for } \nu \text{ odd}. \quad (10)$$

The a_k's are the changes of slope at the break points. If their values are properly determined, they must satisfy certain conditions according to the following reasoning.

Consider the rth integral of 8 which reads

$$F_1^{(\nu+2-r)}(\omega) = \sum a_k[u_{-r}(\omega-\omega_k)+(-1)^\nu u_{-r}(\omega+\omega_k)], \quad (11)$$

in which $u_{-r}(x)$ denotes the singularity function of order $-r$ given by

$$u_{-r}(x) = \frac{x^{r-1}}{(r-1)!} \quad \text{for} \quad x > 0, \quad (12)$$

and equal to zero for $x<0$. From (11) and (12) we get for $\omega=0$

$$F_1^{(\nu+2-r)}(0) = \sum_{k=1}^{n} a_k \frac{\omega_r{}^{r-1}}{(r-1)!}(-1)^\nu, \quad (13)$$

or, if we let $r-1=p$, we may write

$$\sum_{k=1}^{n} a_k \omega_k{}^p = (-1)^\nu p! F_1^{(\nu-p+1)}(0). \quad (14)$$

For $\nu-p+1=0$ this result gives

$$\sum_{k=1}^{n} a_k \omega_k{}^{\nu+1} = (-1)(\nu+1)! F_1(0). \quad (15)$$

The a_k's must satisfy this relation in order to insure zero error in the approximation of $F_1(\omega)$ at $\omega=0$.

Since odd order derivatives of $F_1(\omega)$ must pass through the value zero at $\omega=0$, the result (14) further yields the conditions

$$\sum_{k=1}^{n} a_k \omega_k{}^p = 0 \quad \text{for} \quad \left\{\begin{matrix} p = 0, 2, \cdots \nu = \text{even} \\ p = 1, 3, \cdots \nu = \text{odd} \end{matrix}\right\}. \quad (16)$$

The stated range of p values is seen to follow from the fact that $(\nu-p+1)$ in (14) must be odd, so that p is odd when ν is odd, and even when ν is even. Furthermore, the largest value of p occurs for $\nu-p+1=1$ or for $p=\nu$, and the smallest value occurs for $\nu-p+1=\nu$ when ν is odd[4] and for $\nu-p+1=\nu+1$ when ν is even.

In order that the asymptotic character of the complex spectrum function $F(j\omega)$ may properly be reflected in the behavior of $f(t)$, it is found that additional conditions 16 for p values larger than must be fulfilled by the a_k's. Specifically, if $F(j\omega)$ behaves like $1/(j\omega)^m$ for large ω, then additional conditions (16) for p values defined by $\nu<p<\nu+m$ must be fulfilled.[5] In this connection, it must be remembered, of course, that an $f(t)$ determined from the real part of $F(j\omega)$ must in the absence of any discontinuities at the origin be an even function and can behave for small t only like some even power of t. Hence, only the asymptotic behavior of $F(j\omega)$ for m odd can properly be reflected. This actually is an unimportant matter, but if we insist upon things being just right, we can alternately obtain $f(t)$ from the imaginary part of $F(j\omega)$ when m is even. This alternate approach is also given in footnote reference 4.

It is significant to note that the conditions (15) and (16) will in any actual situation be fulfilled automatically, if the evaluation process for the a_k's is properly done. That is to say, fulfillment of these conditions may be regarded as a check on the accuracy of the numerical work. Fulfillment of the additional conditions (16) for p values larger than ν as called for by asymptotic requirements, however, are not automatic, and involve the manner in which the behavior of the real-part function $F_1(\omega)$ is modified in order to terminate it at a finite frequency. Since this modification permits some freedom of choice, it should not be difficult to dispose of it in such a way as to fulfill the additional conditions (16).

In situations where the function $F_1(\omega)$ is graphically given, and one wishes to apply the piecewise linear approximation to its first or higher derivative, the problem of constructing these derivatives with the required accuracy arises. They obviously cannot be constructed with the usual attitude of regarding the value of the derivative at any point as being equal to the slope of the given function at that point, nor can one use a numerical method based upon tables of finite differences. If we were to construct a second or third derivative by such means and then undo the process through repeated graphical integration, the errors in the derivative functions would be found to cause excessive "drift" error in the integration process, with the result that the original function would not be regained. A set of a_k values based upon a broken-line approximation to

[4] The function for $\nu-p+1=\nu+2$ is not included because it consists of impulses and is zero for $\omega=0$ anyway.

[5] For a detailed consideration of this topic, see "Computational Techniques which Simplify the Correlation Between Steady-State and Transient Response of Filters and Other Networks," Research Laboratory of Electronics, (MIT), Technical Report No. 268, September 2, 1953; or E. A. Guillemin, Proc. Nat'l Electronic Conference, vol. 9, Feb. 1954.

such an inaccurate higher derivative would fail to fulfill the conditions (15) and (16).

Such failure can be avoided if we use a graphical construction process that defines successive ordinates in the derivative function by requiring that the accumulated area under this function from $\omega = -1$ to any particular ω value shall equal the ordinate of the given function at the same point; or that the incremental area under the derivative function for a given $\Delta\omega$ shall equal the ordinate increment in the given function for the same $\Delta\omega$. With this attitude, one can construct derivative functions whose subsequent integration regains the given function with the desired accuracy.

If the piecewise linear approximation is to be applied to the first derivative, one can alternately devise a straightforward method of approximating the given function by parabolic arcs directly, and thus save some manipulative work. Since anyone can figure such things out for himself, space will not be consumed here for its discussion.

As an example of the process of fulfilling the conditions (15) and (16), and of meeting a certain desired asymptotic behavior, suppose we assume that $F(j\omega) \rightarrow 1/(j\omega)^3$ for $(j\omega) \rightarrow \infty$, and ask to construct an $F_1(\omega)$ function consisting of parabolic arcs. Its first derivative is, therefore, piecewise linear. We have $\nu = 1$ and $\nu + m = 4$, so conditions (16) must be satisfied for $p = 1, 3$. If we normalize $F_1(\omega)$ ordinate-wise so that $F_1(0) = 1$, we find for (15) and (16) the conditions

$$\begin{aligned} \sum a_k\omega_k &= 0, \\ \sum a_k\omega_k^2 &= -2, \\ \sum a_k\omega_k^3 &= 0. \end{aligned} \tag{17}$$

The simplest approximation will involve three arcs, and the pertinent a_k values must, in this event, fulfill the equations

$$\begin{aligned} a_1\omega_1 + a_2\omega_2 + a_3 &= 0 \\ a_1\omega_1^2 + a_2\omega_2^2 + a_3 &= -2 \\ a_1\omega_1^3 + a_2\omega_2^3 + a_3 &= 0, \end{aligned} \tag{18}$$

in which the implied normalization of the frequency interval fixes $\omega_3 = 1$. An $F_1(\omega)$ function appropriate to this situation will be constructed as soon as we choose values for the break points ω_1 and ω_2. Suppose we let $\omega_1 = 1/3$, $\omega_2 = 2/3$. Equations (18) then yields

$$a_1 = 45, \qquad a_2 = -36, \qquad a_3 = 9, \tag{19}$$

The resulting real part $F_1(\omega)$ is shown at the top of Fig. 2 and its first, second, and third derivatives are illustrated by the successive pictures below it. The corresponding time function according to (10) reads

$$\begin{aligned} f(t) &= \frac{2}{\pi t^3}\left[45 \sin (t/3) - 36 \sin (2t/3) + 9 \sin t\right] \\ &= \frac{2}{\pi}\left[(t^2/27 - (t^4/729) + \cdots\right] \end{aligned} \tag{20}$$

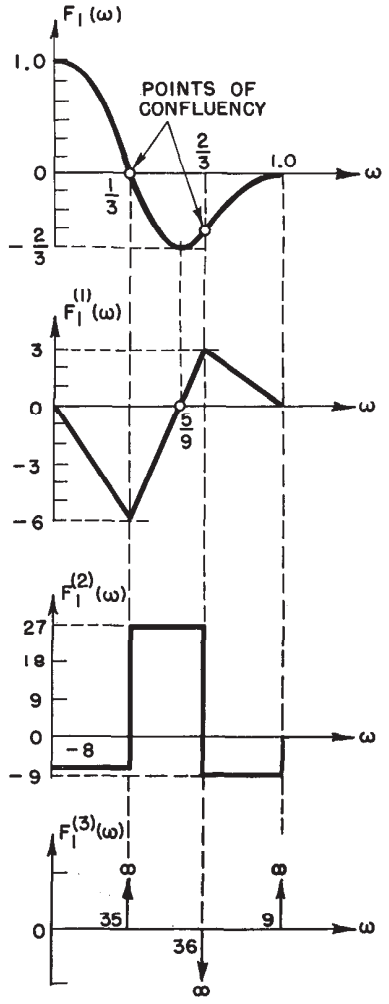

Fig. 2—Real part function represented by parabolic arcs and its successive derivatives.

from which we see that the proper behavior at $t = 0$ is obtained.

Essentially the same methods may be used to find the real and imaginary parts of $F(j\omega)$ from a given time function. Thus, if the *vth* derivative of $f(t)$ is given by a sum of impulses for $t > 0$, then $f(t)$ itself has the representation

$$f(t) = \sum_{k=1}^{n} a_k u_{-\nu}(t - t_k). \tag{21}$$

Since $u_{-\nu}(t)$ has the transform $(j\omega)^{-\nu}$, and delay of a time function by t_k seconds amounts to multiplication by $e^{-j\omega t_k}$ in the frequency domain, we see that the transform of the function (21) is

$$F(j\omega) = \sum_{k=1}^{n} a_k e^{-j\omega t_k} (j\omega)^{-\nu} \tag{22}$$

having the real parts

$$F_1(\omega) = \frac{(-1)^{\nu/2}}{\omega^\nu} \sum_{k=1}^{n} a_k \cos \omega t_k \qquad \text{for } \nu \text{ even} \tag{23}$$

$$F_1(\omega) = \frac{(-1)^{(\nu+1)/2}}{\omega^\nu} \sum_{k=1}^{n} a_k \sin \omega t_k \qquad \text{for } \nu \text{ odd}, \tag{24}$$

and the imaginary parts

$$F_2(\omega) = \frac{(-1)^{(\nu+2)/2}}{\omega^\nu} \sum_{k=1}^{n} a_k \sin \omega t_k \quad \text{for } \nu \text{ even} \tag{25}$$

$$F_2(\omega) = \frac{(-1)^{(\nu+1)/2}}{\omega^\nu} \sum_{k=1}^{n} a_k \cos \omega t_k \quad \text{for } \nu \text{ odd.} \tag{26}$$

II. Evaluation by Analog Computers

There are a great many ways in which a special-purpose analog computer can be constructed for the purpose of evaluating Fourier Transforms. The best method will usually depend on the form in which the data is given, and the accuracy which is required in the result. Here we will discuss representative devices which fall in the following three classes:

1. Devices which mechanize the approximate methods discussed above.
2. Devices which integrate the Fourier Transform (1) directly.
3. Devices which use circuit models to perform the integration.

A. *Approximate Methods*

The approximate method which has been presented in section II depends on replacing the original curve by an approximation whose *nth* derivative is a series of impulses. For the purpose of mechanization it is desirable, although not essential, to make the impulses occur in the function itself, and to require them to be equally spaced. A great deal of accuracy is lost per impulse by these restrictions, but in the analog device it is easier to add additional impulses than to vary their locations. According to the Nyquist sampling theorem, we will lose no information if we take our impulses at a spacing of a half period of the highest frequency contained in our original waveform. On the other hand, such a choice requires very careful filtering to recover the desired information. It is usually preferable to sample more frequently and to filter the output less carefully. The errors involved in this method, and in particular the error in the dc component caused by the finite interval of integration, are covered in the theoretical discussion above. For convenience we repeat (1):

$$f(t) = \frac{2}{\pi} \int_0^\infty F_1(\omega) \cos \omega t d\omega; \quad F(\omega) = \int_0^\infty f(t) \cos \omega t dt. \tag{27}$$

If the function $f(t)$ is replaced by a series of impulses at regular intervals

$$f(t) = f(0)\mu_0(t) + f(\Delta t)\mu_0(t-\Delta t) + f(2\Delta t)\mu_0(t-2\Delta t) \cdots \tag{28}$$

and by the argument used above,

$$F(\omega) = f(0) + f(\Delta t) \cos (\Delta t \omega) + f(2\Delta t) \cos (2\Delta t \omega) + \cdots . \tag{29}$$

Equation (29) can be realized by an analog computer which consists of a number of sinusoidal generators of harmonically related frequencies, and a number of potentiometers which adjust the coefficients of the harmonics. A device which has been used successfully[6] is shown in Fig. 3.

The oscillators shown in Fig. 3 are of the RC type to facilitate exact locking of the higher harmonics. (Oscillators have been synchronized as high as the 80th harmonic.)

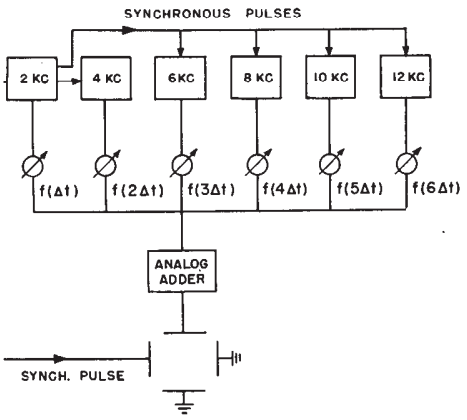

Fig. 3—Analog device for Fourier transforms.

Obvious extensions of this approach are the use of impulse functions for the higher derivatives followed by an analog multiplication by function $1/\omega$, $1/\omega^2$, etc.

In the method which has been discussed above, the function is quantized in the time domain, but the frequency domain response is continuous. A second analog method makes use of quantization in both domains. At some particular value of ω, say $\omega = \omega_1$, (29) gives the following expression for $F(\omega_1)$:

$$F(\omega_1) = f(0) + f(\Delta t) \cos (\Delta t \omega_1) + f(2\Delta t) \cos (2\Delta t \omega_1) + \cdots . \tag{30}$$

Each term in (30) can be realized easily in an analog device. For example, if $f(n\Delta t)$ is given as a voltage, a resistor of conductance $\cos (n\Delta t \omega_1)$ can be bridged across it and the current through it will be

$$i = f(n\Delta t) \cos (n\Delta t \omega_1). \tag{31}$$

A device of this sort to compute a finite number of $F(\omega_n)$ values from a finite number of $f(n\Delta t)$ values, is shown in Fig. 4 on page 248.

If only limited accuracy is required, the type of analog shown in Fig. 4 is extremely satisfactory. It should be noted that the output frequency response is computed almost instantaneously from the given input values of $f(t)$. Thus, Fourier spectrums of charging inputs can be obtained in real time simply by taking the time functions from successive taps on a delay line. For high accuracy this type of analog device is not satisfactory, because of the prohibitive number of resistors required. The use of resistance paper to provide the resistors is a step forward in this direction.

[6] John Petrishen, "Electric computer for Fourier transforms," R.L.E. Quarterly Progress Report, Research Laboratory of Electronics, MIT, June, 1953.

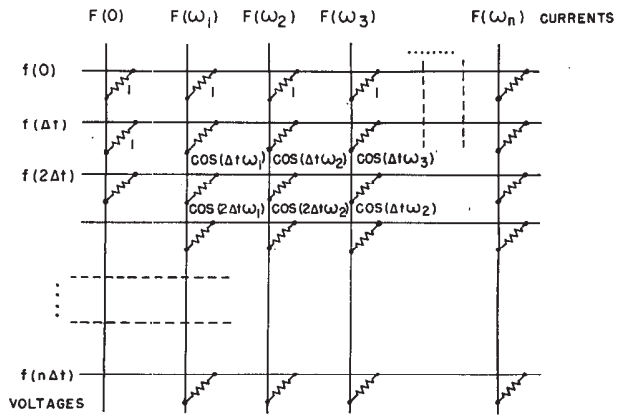

Fig. 4—Fourier transform device with sampling in both domains.

B. *Direct Integration*

Equation (1) can be integrated directly on an analog computer, provided that some way can be found to generate a function of two variables.

$$F_1(\omega) = \int_0^\infty f(t) \cos \omega t dt. \tag{31}$$

The function of two variables which must be obtained is (cos ωt). An early mechanization[7] is shown in Fig. 5. A standard analog computer is used to generate a cosine function whose frequency is slowly changed by varying

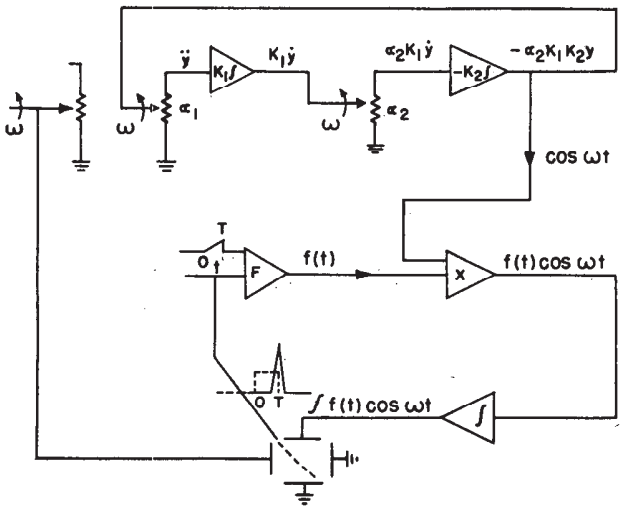

Fig. 5—Fourier transform machine from differential analyzer.

the gain around the loop. The function $f(t)$ is generated in a high-speed computer, multiplied by cos $\omega_n t$ and integrated from $0 \rightarrow T$. During this time, the cosine is essentially constant at the frequency ω_n and the value of the integral at the end of the interval is thus $F(\omega_n)$. The final value is recorded on the scope by a brightening pulse at the end of the interval. The sweep proportional to ω is obtained from a third potentiometer which is synchronized with the two which determine the frequency. The three potentiometers are driven at a slow speed by a motor.

The principal errors in this method arise from the finite interval of integration, and the variation of ω during the integration. The latter error could be removed by using a stepping relay instead of a potentiometer to vary the frequency.

Other types of two-dimensional function generators have been tried. A very good one is based on a cathode-ray tube mask in which the function is proportional to the light transmitted through the mask. Rather than a variable photographic density, it is better to use a variable number of small dots in a given area and to make the density of all the dots the same. If the cathode-ray tube spot diffuses over many dots, the average light intensity will be proportional to the number of dots per unit area. Such a two-dimensional function generator can be used to replace the cosine generator in Fig. 5. Piecewise planar two-dimensional function generators have also been tried, but the cosine function is a difficult one to realize by this technique.

C. *Circuit Models*

The use of a variable-frequency tuned circuit to obtain the frequency response is one of the oldest methods of obtaining a Fourier Transform. The energy at any one frequency is infinitesimal in a function with a continuous spectrum. In order to measure this energy, two artifices can be used. One is a widening of the band that is passed until the necessary amount of energy is available. The second is a repetition of the function over and over at a rate which is slow, relative to the lowest frequency of interest. The Fourier Series is obtained for the repetitive function. The Fourier integral has the form of a continuous curve passing through the values of the harmonics of the Fourier Series. The simplest

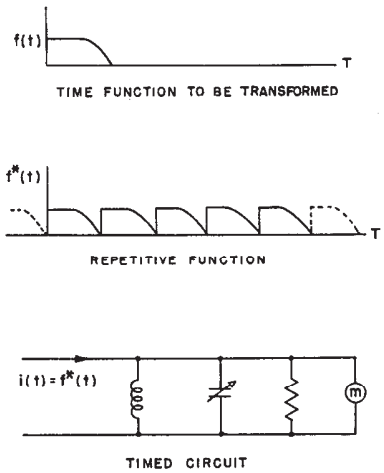

Fig. 6—Fourier analysis device using tuned circuit.

realization is shown in Fig. 6. The function $f(t)$ which is to be transformed is recorded on tape so that it can be repeated endlessly. The repeated wave is fed as a current into the tuned circuit. The circuit is tuned to

[7] Alan B. Macnee, "A High Speed Product Integrator," Technical Report No. 136, Research Laboratory of Electronics, MIT, August, 1949.

the harmonics of the repetition rate and the relative amplitudes of response are obtained. The transform of the voltage across the tuned circuit is

$$V(s) = \frac{1}{C} \frac{F_1(s)s}{s^2 + 2\alpha s + \omega_a^2} \tag{32}$$

where

$$\begin{cases} \alpha = \dfrac{G}{2C} \\ \omega_d^2 = \omega_0^2 - \alpha^2 \\ \omega_0^2 = \dfrac{1}{\sqrt{LC}} \cdot \end{cases}$$

The weighting function in (32) is

$$\frac{j\omega}{(\omega_d^2 - \omega^2) + 2j\omega\alpha} \cdot \tag{33}$$

When α is small, (33) approximates an impulse at $\omega = \omega_d$. If it were a perfect impulse, (32) would become

$$V(j\omega) = \frac{1}{C} F_1(j\omega)\mu_0(\omega - \omega_d), \tag{34}$$

which is an impulse at $\omega = \omega_d$ with a magnitude of $F_1(j\omega)$.

When the weighting function is not a perfect impulse, an average response is obtained in the neighborhood of $\omega = \omega_d$.

An obvious extension of this method is the use of a modulation scheme whereby the signal to be examined is combined with a variable frequency swept signal and a single-tuned circuit is used to obtain the response. When the signal is swept through the circuit, however, it is possible to excite frequency transients. The frequency transients become worse as the circuit Q is raised and as the speed of sweeping the frequency is increased.

A final method of taking Fourier Transforms makes use of the concept of the complex frequency domain. By using active circuits it is possible to construct elements whose transfer functions represent pole and zero positions in the complex plane. These elements can then be cascaded to obtain a transfer function of any desired complexity. A sweep frequency signal is applied to the input and the poles and zeros are adjusted until the steady-state frequency response is identical with the curve which is to be transformed. Then a sharp pulse, approximating an impulse is applied to the circuit and the output time function is measured. It is the transform of the given curve. A second-order transfer function can be produced by an analog computer circuit similar to the oscillator section of Fig. 5. The input is the initial condition and the output is the output voltage of one of the integrators. These circuits have the advantage that the frequency and damping of the poles can be controlled independently and by means of simple potentiometers. Furthermore, they can represent unstable as well as stable circuits over short periods of time.

D. *Summary of Analog Computer Methods for Fourier Transforms*

A few illustrative methods have been suggested for taking Fourier Transforms by analog computers. A great many more have been devised and additional ones can still be invented. The particular type which should be used in a given problem depends on the form of the input data, and the accuracy, dynamic range, and frequency response which are required in the output.

III. Evaluation by Digital Computers

When the inverse Fourier Transform is found by an automatic digital computer, the problem is the evaluation of the integral of (1). In principle, there is no difference between the use of an automatic computer and the use of hand computational techniques; however, the automatic computer has capacity for making many more simple operations than a human computer could make in reasonable time. Because of the effort required to program the computer to solve a problem, it is not practical to modify the procedure with each new $F_1(\omega)$ function. One would program the computer to evaluate the integral of (1) for a sequence of values of t with the program designed to handle all $F_1(\omega)$ functions in the same way. Whereas the hand computational procedures which have been discussed earlier in this paper are tailored to the given $F_1(\omega)$ function with only a few simple terms to evaluate in getting $f(t)$, the automatic computer solution would use a regular quadrature formula such as the rectangular rule, the trapezoidal rule, or Simpson's 1/3 rule for integration.

When the rectangular rule is used, the equivalent of (1) would be given by (35):

$$f(t) = \frac{2\Omega}{\pi} \sum_{k=0}^{n} F_1(k\Omega) \cos k\Omega t, \text{ where} \tag{35}$$

Ω = frequency interval between sample points in the frequency domain.

The value $f(t)$ would be computed at a sequence of values of t. For any other integration rule, the general procedure would be the same. One very important practical problem is how to get the values $F_1(k\Omega)$ and cos $k\Omega t$ into the computer. The values of F_1 would be stored and the values of cos $k\Omega t$ could either be found by referring to a table or by a direct computation. Probably the direct computation method is easier. Since in the course of the integration one would have cos $(k-1)\Omega t$ and cos $(k-2)\Omega t$ and would next need to find cos $k\Omega t$, a very simple computation is possible to relate cos $k\Omega t$ to previous values. The relationship is given in (36):

$$\cos k\Omega t = 2 \cos \Omega t \cos (k - 1)\Omega t - \cos(k - 2)\Omega t. \tag{36}$$

Rather than to store cos $k\Omega t$ for many values of $k\Omega t$, one can compute it as it is needed. This computation

requires considerable accuracy if several cycles of cos $k\Omega t$ are needed unless the t points are chosen so that Ωt is an integral fraction of 2π. One would start the process by knowing cos 0 and cos Ωt for the particular value of t for which (1) was being evaluated. In the computation for cos $2\Omega t$, some errors would be committed because of the finite number length. The error caused in cos $2\Omega t$ tends to cause the later computation for cos $3\Omega t$ to be in error, and so on. The round-off errors in the computation tend to persist. Equation (36) can be looked at as a second-order difference equation whose solution is a cosine sequence. Accordingly, the transient response of the difference equation is an undamped cosine wave and errors committed at one point will not damp out with higher values of k. If Ωt is an integral fraction of 2π, the difference equation would be developed for only one cycle, then it would be exactly repeated. With this procedure the errors would propagate for only 1 cycle.

One would like to obtain an estimate of the errors committed in the digital computer solution. In addition to round-off errors, there are two kinds of errors committed. In the first place, the upper limit of integration turns out to be $(n+1)\Omega$ instead of ∞, and in the second place, the integral is replaced by a finite sum. Replacing the infinite upper limit of the integral by a finite upper limit is the same as putting the transform of $f(t)$ through a unity gain band limited filter of bandwidth $(n+1)\Omega$ before taking the inverse transform. As is well known, this process is the same as convolving the original time function with

$$\frac{\sin (n+1)\Omega t}{(n+1)\Omega t}.$$

Limiting the range of integration thus causes the Gibbs oscillations to appear at discontinuities of the time function. Since the effect of replacing the continuous integral by a finite sum cannot readily be looked upon as a manipulation of $F_1(\omega)$ alone, it is not easy to describe the effect of the numerical integration on $f(t)$ in any general way. In practice, one would perform the integration over the same total frequency range for several successive frequency increments, Ω, and would be satisfied with the quadrature accuracy if halving Ω did not materially affect the time function. The use of a more complicated integration rule than the rectangular rule would not be hard to implement. Probably a rule more complicated than Simpson's 1/3 rule is not justified.

The essential distinction between the use of an automatic digital computer and the techniques for hand calculation is in the lack of tailoring of the method to individual transforms in the automatic computation. The main value in the use of an automatic computer is that one can fairly cheaply and quickly evaluate many transforms-time function pairs. In the use of the automatic computer, one gains insight from the results more than from the procedure.